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An extension of the Slätis theory of the Drude-Coolidge method of measuring the dielectric constant and the coefficient of absorption

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With 3 figures in the text

Abstract

The theory of the Drude-Coolidge method of measuring the dielectric constant and the coefficient of absorption in an UHF-field leads to eqs. (6) and (8). The first is due to Slätis, the second is newly derived. We suggest that they be used in place of the conventional (1), (2) when the so-called "leads effect" must be taken into account. The limit of applicability of eqs. (6), (8) to dielectrics possessing appreciable conductivity is evaluated.

The well-known "second Drude's" method of measuring the dielectric constant and the absorption in the UHF-field occupies a unique position. It has been used frequently and without any major change for more than 50 years since Drude introduced it[1], in spite of its rather limited accuracy. The main reason for this popularity is that, unlike other methods, it requires very small specimens which may be kept in a closed container thus eliminating any danger of contamination or evaporation.

The method consists in using a "Lecher system", i.e. two parallel wires bounded by two bridges, B and B', and a small condenser C put across the wires; see Fig. 1. Bridge B' is movable, allowing the system to be tuned in resonance with a generator (not shown) loosely coupled with it. The condenser contains the dielectric chosen. The dielectric constant and the coefficient of

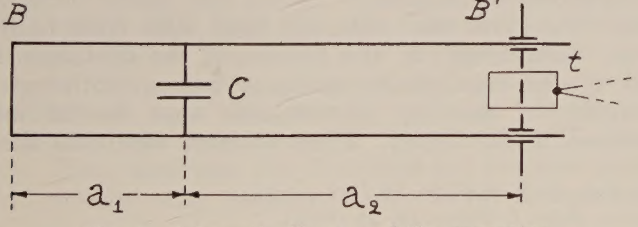

Fig. 1. Schematic diagram of the apparatus.

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[1] For a review of early literature see Hand. d. Phys. *15*, 501 (1927).

absorption can be determined from two resonance curves: one measured when the system is loaded (i.e., when the condenser is filled with a dielectric), another one measured when the system is empty (i.e. condenser either empty or absent). A conventional detector, such as the thermocouple t shown, may be used to measure the resonance curves.

I.

The theory developed by Coolidge[1] and extended for the case of appreciable absorption by Potapenko[2] led to the following equations

$$\delta_0 + \varepsilon\delta = \frac{\lambda}{2}[\cot\alpha a_1 + \cot\alpha a_2] \tag{1}$$

$$\frac{\varkappa}{1-\varkappa^2} = \frac{d\gamma}{4\pi}\left(1+\frac{\delta_0}{\varepsilon\delta}\right)\left[1+\frac{\alpha a_1\dfrac{\sin\alpha a_2}{\sin\alpha a_1}+\alpha a_2\dfrac{\sin\alpha a_1}{\sin\alpha a_2}}{\sin\alpha(a_1+a_2)}\right] \tag{2}$$

$$\delta_0 = 4\pi k_0 \ln\frac{s}{r} \qquad \delta = 4\pi k \ln\frac{s}{r}$$

where ε and $\varkappa$ are the dielectric constant and the coefficient of absorption of the dielectric; λ is the wavelength of the generator in cm; a_1 and a_2 are shown on Fig. 1 and measured in cm; $\alpha = 2\pi/\lambda$; k_0 and k are respectively the ballast capacity of condenser (the part of it not depending on ε) and the working capacity of it (the part depending on ε), both in esu; s is the distance between the centers of parallel wires; r is their radius; $d\gamma = \gamma - \gamma_0$ where γ and γ_0 are the logarithmic damping decrements of the loaded and of the empty system respectively.[3]

A condenser put across the wires represents a capacitive load, as long as the inductive effect of its leads is small (see later). In such case $a_1 + a_2$ must be smaller than $\lambda/2$, or

$$\frac{\lambda}{2} - a_1 - a_2 = d \tag{3}$$

where d is the so-called "displacement" of the bridge B. Under the capacitive load d must, therefore, be positive.

Eqs. (1) and (2), show that their left hand sides must be independent of a_1 and a_2, i.e., independent of the position of the condenser in the system. The fulfilment of this requirement, necessary but not sufficient for the proof of the applicability of eqs. (1), (2), depends upon the fulfilment of several conditions assumed in the theory. Three of these conditions are of particular

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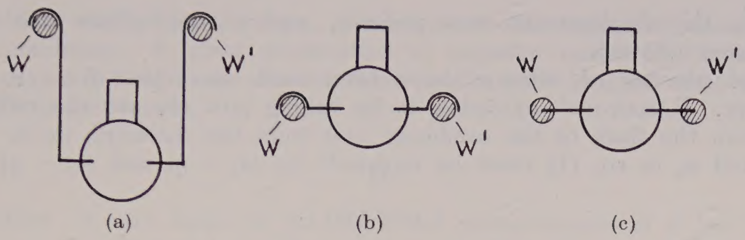
[1] W. D. Coolidge, Ann. d. Phys., *69*, 125 (1899).

[2] G. Potapenko, Zeits. f. Phys., *20*, 21 (1923).

[3] In the case of electrolytes, $d\gamma = \gamma - m\gamma$ where m is a factor varying with conductivity. It may be calculated in the manner shown by B. K. Maibaum, J. exp. theor. Phys. *14*, 448, 501 (1944).



(a) (b) (c)

Fig. 2. Condensers.

importance. The first one governs the choice of the condenser. The capacity C of it must be a linear function of ε, i.e.

$$C = k_0 + \varepsilon k \tag{4}$$

In reality eq. (4) is just an approximation. Its correctness depends upon the shape of condenser. When liquids are studied, the condenser is usually shaped, as shown on Fig. 2 a — "long leads"; or Fig. 2 b — "short leads"; w_1, w_2 are the parallel wires.

The leads protrude inside the glass vessel, and sometimes they are terminated by small discs. Eq. (4) holds fairly well when the inside surface of the vessel follows the electric lines of force in the liquid. The second condition refers to the choice of a_1. It can be shown that eq. (2) is nearly correct when $a_1 < a_2$, but both eqs. (1), (2) require that a_1 must not be smaller than a certain limit. This limit must be determined experimentally. It depends mainly upon the capacity of the condenser with respect to the bridge B. This capacity is not taken into account in eqs. (1), (2) and its effect may introduce a noticeable error when a_1 becomes small. The third condition requires that the losses in the system remain low. In practice it means that the static conductivity of dielectric must not exceed a certain limit. This limit depends upon the design of the condenser, in particular upon the length of its leads, as it will be shown later.

The limitations of eqs. (1), (2) can be demonstrated by comparing the results of measurements made under different conditions. The best example we know is furnished by two sets of measurements on alcohols; the older one[1] obtained using a condenser of the "long leads" type, Fig. 2 a, the later one[2] obtained with a condenser of the "short leads" type, Fig. 2 b. In both cases the same frequency range, and Lecher systems of almost identical dimensions were used. The results show that when the absorption is relatively low, as in Methylalcohol at $\lambda > 50$ cm, the ε and $\varkappa$ of the two sets are in a good agreement. Contrary to this, in cases of appreciable absorption ($\varkappa > 0.15$) the results depended entirely upon the type of condenser. In the case of Propylalcohol at $\lambda \sim 30$ cm, with a condenser of "long leads" type, the displacement d was negative. This, were eqs. (1), (2) correct and complete could only happen if the dielectric constant were negative. When a "short leads" type condenser was used, the displacement was positive, and ε was in line with what the dipole theory allowed.

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[1] G. Potapenko, Proc. Inst. Phys. a. Crystallogr., U. of Moscow, *6*, 1 (1926).

[2] E. Keutner and G. Potapenko, Phys. Zeits. *40*, 100 (1939).

A large number of attemps have been made since that time, to improve the theory. ROMANOFF[1] extended it by taking into account the reflection of waves from the flask of the condenser and from the dielectric in it. In such case a_1 and a_2 in eq. (1) must be replaced[2] by $(a_1 - \eta)$ and $(a_2 - \eta)$

$$\eta = f(\varepsilon) \tag{5}$$

which must be determined experimentally. POTAPENKO[3] took into account the potential drop along the leads of condenser. The effect of this drop must increase with the length of leads, and may become large when their length becomes noticeable compared with $(a_1 + a_2)/2$. The potential drop across the dielectric decreases with absorption. On account of this the effect of the potential drop along the leads of condenser must increase with absorption. This conclusion agrees well with what the measurements on alcohols referred to above have shown. The effect of the leads can be easily minimized by making them short.

SLÄTIS[4] extended the theory by taking into account both the potential drop along the leads and the inductance effect due to eddy currents in them. For the case of short leads the theory shows

$$\delta_0 + \varepsilon\delta + 2\pi\vartheta = \frac{\lambda}{2}\left[\cot \alpha\,(a_1 - \vartheta) + \cot \alpha\,(a_2 - \vartheta)\right] \tag{6}$$

$$\vartheta = \frac{L_{12}^2}{4\,L_{22}\ln\dfrac{s}{r}} \tag{7}$$

where L_{12} is the mutual inductance between each condenser lead and its neighboring Lecher wire; L_{22} is the self-inductance of the leads; s, r are the same as in eqs. (1), (2). The term $2\pi\vartheta$ being constant it could be combined with δ_0. This makes eq. (6) equivalent to Romanoff's equation except for the meaning of constants η, δ_0 and ϑ. Physical meaning of the constant ϑ is defined by eq. (7), but on account of the difficulty of determining L_{12}, L_{22} it cannot always be calculated accurately. As a result, SLÄTIS' constant ϑ in practice is just an experimental constant, as ROMANOFF's constant η is. The only real difference between them is in the fact that η depends slightly upon ε while ϑ is a function of the dimensions of the apparatus only. Later ROMANOFF[5] derived another type of equation to take into account the effect of leads on measured ε values, and MALOV[6] made an attempt to interpret the effect by considering the inductive load which they produce, in addition to the capacitive load produced by the condenser. A clear explanation of negative values of d can be given in this way. We will not discuss their results here because what they yield does not replace eq. (6).

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[1] W. ROMANOFF, Phys. Zeits. S. U., *9*, 362 (1936).

[2] Provided $\sin \alpha\eta \sin \alpha\,(a_1 + a_2 - \eta)/\sin \alpha\,a_1 \sin \alpha\,a_2$ is small compared to 1. This requirement is usually satisfied.

[3] G. POTAPENKO, Proc. Int. Congr. on Sh. Waves, Vienna (1937), 133.

[4] H. SLÄTIS, Ann. d. Phys. *32*, 734 (1938). The theory of the effect of potential drop along leads for the special case $a_1 = a_2$ have been published by him in Acta Ac. Aboensis, *9*, No. 4 (1936).

[5] W. ROMANOFF, J. Exp. Theor. Phys., *8*, 328 (1938).

[6] N. MALOV, J. Exp. Theor. Phys., *9*, 864 (1939).

II.

The effect of the leads on the results of measurements of $\varkappa$ has also been investigated by SLÄTIS[1] but only for the case when the maxima of resonance curves are used in place of full curves. The accuracy of determination of both ε and $\varkappa$ can be considerably increased when full resonance curves are measured. First, the errors due to the fluctuation of the output of the generator can be reduced many times in this way; second, the use of chord midpoints from a full resonance curve provides a much better determination of the position of the maximum, especially when the absorption is high and the resonance curves are flat.

SLÄTIS' theory may be easily extended to the case when full resonance curves are measured. It yields[2]

$$\frac{\varkappa}{1-\varkappa^2}=\frac{d\,\gamma}{4\pi}\,\frac{\delta_0+\varepsilon\,\delta+2\pi\vartheta}{\varepsilon\,\delta}\cdot$$
$$\cdot\left[1+a\,\frac{(a_1-\vartheta)\,\dfrac{\sin a\,(a_2-\vartheta)}{\sin a\,(a_1-\vartheta)}+(a_2-\vartheta)\,\dfrac{\sin a\,(a_1-\vartheta)}{\sin a\,(a_2-\vartheta)}}{\sin a\,(a_1+a_2-2\,\vartheta)}\right]. \qquad (8)$$

Eq. (8) is a counterpart of eq. (2). It shows that the effect of the leads can be accounted for by substituting in eq. (2) $(a_1-\vartheta)$ and $(a_2-\vartheta)$ instead of a_1 and a_2, and by adding the term $2\pi\vartheta$ as shown.

III.

Eqs. (6), (8) must be used in place of the old eqs. (1), (2) when the effect of the leads, as represented by the constant ϑ, cannot be neglected.

The question remains as to how large ϑ is in practice. When a "long leads" type condenser is used, and when $\lambda < 50$ cm and when a_1 is small as in the case of an appreciable absorption, then ϑ cannot be neglected as the results on alcohols referred to above show. SLÄTIS in his measurements[1] on Propylalcohol, made using a "short leads" type condenser at $\lambda \sim 70$ cm assumed $\vartheta = 0$. Accordingly he used eq. (1) in place of his own eq. (6) to compute ε.

In our experiments we used a condenser with short leads, the ends of which were soldered into holes in the Lecher wires; see Fig. 2 c. The spherical glass flask was 1.0 cm in diameter. The leads were platinum wires 0.05 cm in diameter, projecting about 0.01 cm into the flask. The Lecher system was of conventional design, of copper wires 0.20 cm in diameter, spaced 2.0 cm between the centers. The detector was a vacuum thermo-couple loosely coupled with the system by a small rectangular loop, in the plane of Lecher wires, protuding 0.6 cm into the system through holes in the movable bridge B_1; see Fig. 1. The relative position of bridge B_1 was measured with a dial indicator, with an accuracy of better than 0.001 cm. The generator of high stability, previously described[1], was built using a GE-FP-126 tube; it was tuned to $\lambda \sim 25.6$ cm and this did not vary by more than 0.01 cm during several hours of continuous work.

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[1] H. SLÄTIS, Ann. d. Phys. *36*, 397 (1939).

[2] For derivation see Appendix; for details see D. WHEELER, Ph. D. Thesis, California Institute of Technology (1947).

To determine ϑ we measured a_2 while varying a_1. Each pair of a_1, a_2 values[2] was then used to compute the right hand side of eq. (6) using several trial values of ϑ. By linear interpolation, the value of ϑ which gave the least mean square deviation was determined. In this way the value $\vartheta = 0.011$ cm has been obtained. The effect of this correction on the constancy of $\delta_0 + \varepsilon \delta$ can be seen from the example given in Table I.

Table I

Benzene; $\lambda = 25.550$ cm

a_1 (cm)	a_2 (cm)	$\delta_0 + \varepsilon \delta$	
		$\vartheta = 0$	$\vartheta = 0.011$
1.180	11.571	0.902	1.696
1.580	11.135	1.262	1.700
2.080	10.578	1.451	1.692
2.580	10.009	1.564	1.715
3.080	9.444	1.569	1.673
3.580	8.870	1.613	1.688
4.080	8.317	1.578	1.647

The ϑ we found is small, but it cannot be neglected when ε and $\varkappa$ need to be determined accurately.[3]

In our measurements we used air, benzene and propionic acid as dielectrics, and we could not notice any difference in value of ϑ obtained. This means, we could not notice any dependence of ϑ on ε as eq. (5) requires were the effect of reflection from the dielectric in the condenser noticeable. This agrees well with FRADKINA[4] who, by using a set-up especially designed to exaggerate the inaccuracy of eqs. (1), (2), came to the conclusion that the effect of the leads must predominate over the effect of reflection.

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[1] G. POTAPENKO, Phys. Rev. *32*, 625, 638 (1932).

[2] To each measured a_1, a_2 value, accurate to about 0.001 cm, was added 0.055 cm as the "bridge contraction" correction determined in the conventional manner from the difference between $\lambda/2$ and the distance L between the bridges measured when the condonser was removed from the system.

[3] Our ϑ value casts no doubt on SLÄTIS' assumption of $\vartheta = 0$, because in his case the wires of the Lecher system were closely spaced ($s = 1.1$ cm), and the leads were shorter, thus reducing ϑ. A larger spacing s we used has an advantage: it allows one to reduce the working capacity of condenser, and through this to extend the measurements to dielectrics of higher absorption without overloading the system.

[4] E. M. FRADKINA, J. exp. theor. Phys., *9*, 1379 (1939).

IV.

The applicability of eqs. (6), (8) to cases when the static conductivity of dielectrics is appreciable, depends greatly upon the magnitude of the absorption due to reasons other than conductivity. This means that the question should be answered in terms of limiting values of $\varkappa$.

The effect of conductivity on the results of DRUDE's method has been studied on several occasions.[1] From the theory of SOSINSKI and DMITRIEFF it follows that this effect may be neglected when

$$\frac{\sin^2 a \frac{\Delta}{2}}{1 + \sin^2 a \frac{\Delta}{2}} \ll 1 .$$

Here Δ is the width of resonance curves at its "half-height" point. This width, when the dielectric is given and the working capacity of the condenser is reduced to the minimum, may be decreased by decreasing a_1. Due to this, in practice the limit of applicability of eqs. (6), (8) depends upon the limit of a_1, discussed earlier. As our experience shows, consistent results may be obtained when $\varkappa$ does not exceed $\varkappa \sim 0.4$. This may be considered as the limit beyond which eqs. (6), (8) should not be applied without very careful checking. At $\lambda = 25$ cm such $\varkappa$ roughly corresponds to the absorption in a 3 % NaCl solution ($\sigma_{st} = 0.04\ \Omega^{-1}\ \mathrm{cm}^{-1}$). In this region of σ_{st} and λ, the absorption due to conductivity changes approximately proportionally with λ. This means that by decreasing λ one can extend the region of applicability of eq. (6), (8) but it is obvious that this possibility is quite limited. Due to this, when electrolytes of higher concentration must be studied, some other method of evaluating ε and $\varkappa$ must be chosen. The theory developed by Maibaum (l. c.) takes advantage of the inductive load which long leads produce, and of negative displacement d, to extend the method to strong electrolytes. But the problem is complicated, and more experiments are needed before it will be clear how far this extension leads.

Appendix

Eq. (8) may be derived as follows: consider a Lecher system, Fig. 3, bridged by a condenser of capacity C at $z = 0$ and bounded by two perfectly conducting bridges at $z = -a_1$ and $z = a_2$. The effect of the condenser leads, as SLÄTIS has shown, may be taken into account by assuming that for a small region near the condenser, $|z| < b$, the propagation constant of the system is changed, so that in this region the wavelength of the waves on the wire will be $\lambda' = \lambda / p$ where λ is the wavelength outside this region and $p < 1$.

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[1] N. MALOV, Phys. Zeits., S. U. *12*, 111 (1937); S. SOSINSKI and V. DMITRIEFF, J. Exp. Theor. Phys., *8*, 1384 (1938); B. MAIBAUM (l. c.).

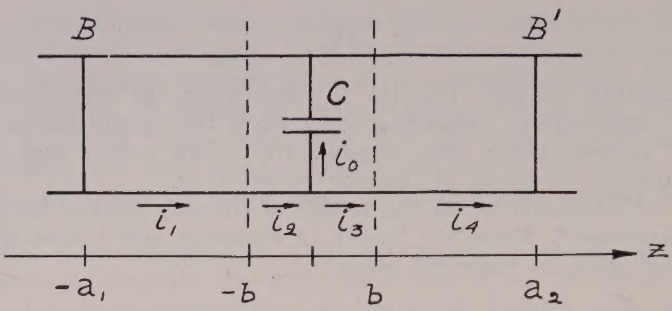

Fig. 3. Currents in a Lecher system.

The current in any section of the Lecher system is given by

$$i_n = A_n e^{\xi\left(\frac{t}{T} - \frac{z}{\lambda_n}\right)} + B_n e^{\xi\left(\frac{t}{T} + \frac{x}{\lambda_n}\right)} \tag{9}$$

where

$$n = 1, 2, 3, 4;\ \lambda_1 = \lambda_4 = \lambda;\ \lambda_2 = \lambda_3 = \lambda / p$$

and

$$\xi = -\gamma + 2\pi i \tag{10}$$

where γ is the logarithmic damping decrement of the system.
The boundary conditions to be satisfied are:

$$\begin{aligned}
&\text{at } z = -a_1 : \frac{\partial i_1}{\partial z} = 0 \\
&z = -b : i_1 = i_2 \qquad \text{and} \quad \frac{\partial i_1}{\partial z} = \frac{\partial i_2}{\partial z} \\
&z = 0 \quad : i_2 - i_3 = i_0 \quad \text{»} \quad \frac{\partial i_2}{\partial z} = \frac{\partial i_3}{\partial z} \\
&z = b \quad : i_3 = i_4 \quad \text{»} \quad \frac{\partial i_3}{\partial z} = \frac{\partial i_4}{\partial z} \\
&z = a_2 \quad : \frac{\partial i_4}{\partial z} = 0 .
\end{aligned} \tag{11}$$

For the current through the condenser we have

$$i_0 = C \frac{\partial V}{\partial t} + \frac{V}{W} \tag{12}$$

where V is the potential difference between the condenser plates and W is the resistance between the plates. Substituting this into the first boundary condition at $z=0$, and making use of the relations $V = 4\,q\,c^2 \ln s/r$, $\frac{\partial q}{\partial t} = -\frac{\partial i}{\partial z}$, where q is the charge per unit length of the system, we obtain the new boundary condition at $z=0$

$$\frac{\partial i_2}{\partial t} - \frac{\partial i_3}{\partial t} = -\,4\,c^2 \ln \frac{s}{r}\left[C\,\frac{\partial^2 i_3}{\partial t\,\partial z} + \frac{1}{W}\,\frac{\partial i_3}{\partial z}\right].$$

Substituting eq. (9) into the boundary conditions and making the abbreviation

$$l = 4\,c^2 \ln \frac{s}{r}\left[C + \frac{T}{\xi W}\right] \tag{13}$$

give us the eight equations

$$\begin{aligned}
&-A_1 e^{\frac{\xi a_1}{\lambda}} + B_1 e^{\frac{\xi a_1}{\lambda}} = 0\\
&A_1 e^{\frac{\xi b}{\lambda}} + B_1 e^{-\frac{\xi b}{\lambda}} = A_2 e^{\frac{\xi b p}{\lambda}} + B_2 e^{-\frac{\xi b p}{\lambda}}\\
&-A_1 e^{\frac{\xi b}{\lambda}} + B_1 e^{-\frac{\xi b}{\lambda}} = -\,p\,A_2 e^{\frac{\xi b p}{\lambda}} + p\,B_2 e^{-\frac{\xi b p}{\lambda}}\\
&-A_2 + B_2 = -A_3 + B_3\\
&A_2 + B_2 - A_3 - B_3 = l\,\frac{\xi p}{\lambda}\,(A_3 - B_3)\\
&A_3 e^{-\frac{\xi b p}{\lambda}} + B_3 e^{\frac{\xi b p}{\lambda}} = A_4 e^{-\frac{\xi b}{\lambda}} + B_4 e^{\frac{\xi b}{\lambda}}\\
&-\,p\,A_3 e^{-\frac{\xi b p}{\lambda}} + p\,B_3 e^{\frac{\xi b p}{\lambda}} = -A_4 e^{-\frac{\xi b}{\lambda}} + B_4 e^{\frac{\xi b}{\lambda}}\\
&-A_4 e^{-\frac{\xi a_2}{\lambda}} + B_4 e^{\frac{\xi a_2}{\lambda}} = 0\,.
\end{aligned}$$

Eliminating the A's and B's from these equations, and solving for $\frac{l\xi}{\lambda}$ gives

$$\begin{aligned}
\frac{l\xi}{\lambda} = i\Biggl\{&\frac{\frac{1}{p}\cot\left(-\frac{i\xi b p}{\lambda}\right)\cot\left[-\frac{i\xi}{\lambda}(a_1-b)\right]-\frac{1}{p^2}}{\cot\left[-\frac{i\xi}{\lambda}(a_1-b)\right]+\frac{1}{p}\cot\left(-\frac{i\xi b p}{\lambda}\right)}\,+\\
&+\frac{\frac{1}{p}\cot\left(-\frac{i\xi b p}{\lambda}\right)\cot\left[-\frac{i\xi}{\lambda}(a_2-b)\right]-\frac{1}{p^2}}{\cot\left[-\frac{i\xi}{\lambda}(a_2-b)\right]+\frac{1}{p}\cot\left(-\frac{i\xi b p}{\lambda}\right)}\Biggr\}.
\end{aligned} \tag{14}$$

The effect of the leads should be small so that we may consider $\frac{\xi b}{\lambda} \ll 1.$ Then expanding the right hand side of eq. (14) in powers of $\frac{\xi b}{\lambda}$, and neglecting powers higher than the first gives

$$\begin{aligned}\frac{l\xi}{\lambda} = i\Biggl\{\cot\left(-\frac{i\xi a_1}{\lambda}\right) + \cot\left(-\frac{i\xi a_2}{\lambda}\right) - \\ - i\left[\csc^2\left(-\frac{i\xi a_1}{\lambda}\right) + \csc^2\left(-\frac{i\xi a_2}{\lambda}\right)\right]\frac{\xi b\,(1-p^2)}{\lambda}\Biggr\} - \frac{2\xi b\,(1-p^2)}{\lambda}.\end{aligned} \tag{15}$$

But the portion of eq. (15) in curly brackets is just the first term of the expansion of

$$\cot\left\{-\frac{i\xi}{\lambda}[a_1 - b\,(1-p^2)]\right\} + \cot\left\{-\frac{i\xi}{\lambda}[a_2 - b\,(1-p^2)]\right\}$$

in powers of $\frac{\xi b\,(1-p^2)}{\lambda}$, so that, writing $b\,(1-p^2) = \vartheta$, we have

$$\frac{l\xi}{\lambda} = i\left\{\cot\left[-\frac{i\xi}{\lambda}(a_1 - \vartheta)\right] + \cot\left[-\frac{i\xi}{\lambda}(a_2 - \vartheta)\right]\right\} - \frac{2\vartheta\xi}{\lambda}. \tag{16}$$

Substituting for l and ξ from (13) and (10) we obtain

$$\begin{aligned}-\frac{4c^2C\gamma\ln\frac{s}{r}}{\lambda} - \frac{2\vartheta\gamma}{\lambda} + \frac{4c^2T\ln\frac{s}{r}}{W\lambda} + i\left[\frac{8\pi c^2C\ln\frac{s}{r}}{\lambda} + \frac{4\pi\vartheta}{\lambda}\right] = \\ = i\left\{\cot\left[\frac{2\pi}{\lambda}(a_1-\vartheta) + \frac{i\gamma}{\lambda}(a_1-\vartheta)\right] + \cot\left[\frac{2\pi}{\lambda}(a_2-\vartheta) + \frac{i\gamma}{\lambda}(a_2-\vartheta)\right]\right\}.\end{aligned} \tag{17}$$

Provided the absorption in the system is small, we can expand this in powers of γ and neglect powers higher than the first. Doing so, and separating real and imaginary parts in the result, gives

$$\frac{8\pi c^2C\ln\frac{s}{r} + 4\pi\vartheta}{\lambda} = \cot\frac{2\pi}{\lambda}(a_1-\vartheta) + \cot\frac{2\pi}{\lambda}(a_2-\vartheta) \tag{18}$$

$$\begin{aligned}-\frac{4c^2C\gamma\ln\frac{s}{r} - 2\vartheta\gamma + \frac{4c^2T}{W}\ln\frac{s}{r}}{\lambda} = \\ = \frac{\frac{2\gamma}{\lambda}(a_1-\vartheta)\cot\frac{2\pi}{\lambda}(a_1-\vartheta)}{\sin\frac{4\pi}{\lambda}(a_1-\vartheta)} + \frac{\frac{2\gamma}{\lambda}(a_2-\vartheta)\cot\frac{2\pi}{\lambda}(a_2-\vartheta)}{\sin\frac{4\pi}{\lambda}(a_2-\vartheta)}.\end{aligned} \tag{19}$$

Setting $4\pi c^2 C \ln \frac{s}{r} = \delta_0 + \varepsilon\delta$ and $\frac{2\pi}{\lambda} = a$, eq. (18) becomes

$$\delta_0 + \varepsilon\delta + 2\pi\vartheta = \frac{\lambda}{2}[\cot a(a_1 - \vartheta) + \cot a(a_2 - \vartheta)]$$

which coincides with eq. (6).

Taking the ratio of (18) to (19), and making use of the relation

$$\frac{T}{WC} = \frac{4\pi\varepsilon\delta\varkappa}{(\delta_0 + \varepsilon\delta)(1 - \varkappa^2)}$$

we obtain finally

$$\frac{\varkappa}{1 - \varkappa^2} = \frac{d\gamma}{4\pi}\frac{\delta_0 + \varepsilon\delta + 2\pi\vartheta}{\varepsilon\delta} \cdot \left[1 + \frac{a(a_1 - \vartheta)\frac{\sin a(a_2 - \vartheta)}{\sin a(a_1 - \vartheta)} + a(a_2 - \vartheta)\frac{\sin a(a_1 - \vartheta)}{\sin a(a_2 - \vartheta)}}{\sin a(a_1 + a_2 - 2\vartheta)}\right].$$

California Institute of Technology, Pasadena, Cal.

Tryckt den 17 mars 1950

Uppsala 1950. Almqvist & Wiksells Boktryckeri AB